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Mathieu, M., & Young, M. (2015). Spectral isometries into commutative Banach algebras. In K. Jarosz (Ed.), *Contemporary Mathematics* (Vol. 645, pp. 217-222). (Contemporary Mathematics; Vol. 645). American Mathematical Society. <https://doi.org/10.1090/conm/645>

**Published in:**  
Contemporary Mathematics

**Document Version:**  
Peer reviewed version

**Queen's University Belfast - Research Portal:**  
[Link to publication record in Queen's University Belfast Research Portal](#)

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# SPECTRAL ISOMETRIES INTO COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. We determine the structure of spectral isometries between unital Banach algebras under the hypothesis that the codomain is commutative.

## 1. INTRODUCTION

Spectral isometries, that is, spectral radius-preserving linear mappings, are the non-selfadjoint analogues of isometries between unital  $C^*$ -algebras. Every Jordan isomorphism preserves the spectrum of each element (of the domain), hence the spectral radius. Under the assumption that it is selfadjoint (that is, maps selfadjoint elements onto selfadjoint elements), it is an isometry. Kadison, in 1951, proved the converse and established a non-commutative generalization of the classical Banach–Stone theorem: Every unital surjective isometry between unital  $C^*$ -algebras is a Jordan  $*$ -isomorphism [5]; thus, a self-adjoint spectral isometry. Conversely, every unital surjective spectral isometry which is selfadjoint must be an isometry, an easy consequence of the Russo–Dye theorem. This, amongst others, led to the conjecture that every unital surjective spectral isometry between unital  $C^*$ -algebras is a Jordan isomorphism, see [9], and for a more in-depth discussion of this interplay, [7].

As it stands, the above conjecture is still open though there has been substantial progress towards it. It has been observed, see in particular [10], that the behaviour on commutative subalgebras is vital for the conjecture to hold. Moreover, under additional hypotheses, the conjecture has even been verified for certain Banach algebras; see, e.g., [3] and [1]. This motivated us to re-visit the situation for commutative Banach algebras and to fill in some loose ends in the literature. It has been known for some time that a unital surjective spectral isometry between commutative unital semisimple Banach algebras is an algebra isomorphism; this is Nagasawa’s theorem, see, e.g., [2, Theorem 4.1.17]. What about, however, non-unital or non-surjective spectral isometries in this setting? The present note intends to answer these questions by a unified method.

## 2. NON-UNITAL AND NON-SURJECTIVE SPECTRAL ISOMETRIES

Throughout this paper,  $A$  and  $B$  will denote unital complex Banach algebras, and we shall generally be following the notation in [6]. The (Jacobson) radical of  $A$  is  $\text{rad}(A)$  and  $Z(A)$  stands for the centre of  $A$ .

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2000 *Mathematics Subject Classification.* Primary 47A65; Secondary 46H99, 46J10, 47A10, 47B48.

*Key words and phrases.* Spectral isometries, Jordan isomorphisms, commutative Banach algebras.

Some of the results in this paper were obtained in the second-named author’s PhD thesis written under the supervision of the first-named author.

Let  $T: A \rightarrow B$  be a *spectral isometry*, that is, a linear mapping satisfying  $r(Tx) = r(x)$  for all  $x \in A$ , where  $r(\cdot)$  denotes the spectral radius. It is well known that, if  $T$  is surjective,  $\text{Trad}(A) = \text{rad}(B)$ ; see [9, Proposition 2.11] or [10, Lemma 2.1]. Therefore, by passing to the quotient Banach algebras  $A/\text{rad}(A)$  and  $B/\text{rad}(B)$ , we obtain a canonically induced spectral isometry between semisimple Banach algebras. If  $T$  is not assumed to be surjective but  $B$  is commutative then, since  $\text{rad}(B)$  coincides with the set of all quasi-nilpotent elements in  $B$ , we still have  $\text{Trad}(A) \subseteq \text{rad}(B)$  and the same argument applies. As a result, we shall henceforth assume that our Banach algebras are semisimple (instead of formulating the results “modulo the radical”).

Suppose  $T$  is a surjective spectral isometry. Then  $TZ(A) = Z(B)$  [9, Proposition 4.3], a fact that turned out to be very useful in the non-commutative setting. If  $T$  is not surjective, once again the assumption that  $B$  is commutative will prove itself to be expedient.

Our approach exploits the close relationship between spectral isometries on semisimple commutative Banach algebras and isometries on Banach function algebras; on the latter, there is a vast literature, see, e.g., [4]. The main tool will be a version of Novinger’s theorem and a consequence of it which was originally obtained by deLeeuw, Rudin and Wermer. For convenience, we will formulate this in one result. Recall first that the *Choquet boundary*  $\text{ch}(E)$  of a linear space  $E$  of continuous functions on a compact Hausdorff space  $X$  is defined as

$$\text{ch}(E) = \{t \in X \mid \epsilon_t \text{ is an extreme point of } E_1^*\},$$

where  $E_1^*$  denotes the dual unit ball and  $\epsilon_t$  is the point evaluation at  $t$ .

**Theorem 2.1** ([4], Theorem 2.3.10 and Corollary 2.3.16). *Let  $X$  and  $Y$  be compact Hausdorff spaces and denote by  $C(X)$  and  $C(Y)$  the Banach algebras of continuous complex-valued functions on  $X$  and  $Y$ , respectively. Let  $E \subseteq C(X)$  be a subspace which separates the points of  $X$  and contains the constant functions. Suppose  $T$  is a linear isometry from  $E$  onto a subspace  $F \subseteq C(Y)$ . Then there exist a unimodular function  $h \in C(Y)$  and a continuous function  $\varphi$  from  $\text{ch}(F)$  onto  $\text{ch}(E)$  such that*

$$Tf(t) = h(t) f(\varphi(t)) \quad \text{for all } f \in E \text{ and } t \in \text{ch}(F). \quad (2.1)$$

*If, moreover,  $E$  and  $F$  are unital subalgebras then  $T_1$  defined by  $T_1f = \bar{h}Tf$ ,  $f \in E$  is an algebra isomorphism from  $E$  onto  $F$ .*

In particular, if the isometry  $T$  is *unital*, that is,  $T1 = 1$ ,  $T$  will be an algebra isomorphism from  $E$  onto  $F$  if and only if  $F$  is a subalgebra of  $C(Y)$ . In general, the image of a unital isometry defined on a subalgebra of  $C(X)$  need not be a subalgebra of  $C(Y)$ . Since this fact partly motivates our paper, we recall one of the well-known examples.

*Example 2.2* (McDonald, see [4], Example 2.3.17). Let  $\varphi_1, \varphi_2$  be continuous functions from the compact Hausdorff space  $Y$  into the compact Hausdorff space  $X$ . Define  $T: C(X) \rightarrow C(Y)$  by  $Tf(t) = \frac{1}{2}(f(\varphi_1(t)) + f(\varphi_2(t)))$ ,  $t \in Y$ . Let  $\Gamma = \{t \in Y \mid \varphi_1(t) = \varphi_2(t)\}$ . If  $\varphi_1(\Gamma) = X$  then  $T$  is a unital isometry. However,  $F = \text{im } T$  is not a subalgebra of  $C(Y)$  in general since  $\text{ch}(F) = \Gamma$  which may be smaller than  $Y$ . Indeed,

for  $t \in Y \setminus \Gamma$  take  $f \in C(X)$  such that  $f(\varphi_1(t)) = 1$  and  $f(\varphi_2(t)) = 0$ . Then

$$(TfT(1-f))(t) = -\frac{1}{4} \quad \text{whereas} \quad T(f(1-f))(t) = 0.$$

The Choquet boundary of a subspace  $F \subseteq C(Y)$  is always a boundary for  $F$  in the sense that, for each  $g \in F$ , there is  $t \in \text{ch}(F)$  such that  $\|g\| = |g(t)|$  (Phelps' theorem, see, e.g., [4, Theorem 2.3.8]). The above example illustrates nicely the fact that the image  $F$  of an isometry will only be an algebra if  $\text{ch}(F)$  is a boundary for the algebra generated by  $F$ , which is also the core of the argument to deduce the second part of Theorem 2.1 from the first.

The connection between spectral isometries on commutative Banach algebras and isometries on function algebras is of course made via Gelfand theory, but this seems not to have been exploited so far. For a unital commutative semisimple Banach algebra  $A$  we let  $\Delta(A)$  denote its structure space, that is, the space of multiplicative linear functionals on  $A$  endowed with the weak\* topology or, equivalently, the maximal ideal space of  $A$  with the hull-kernel topology. See [6, Chapter 2]. Recall that  $\Delta(A)$  is a compact Hausdorff space.

We shall use  $\Gamma_A: A \rightarrow C(\Delta(A))$  to denote the Gelfand transformation of  $A$  and abbreviate the image of  $a \in A$  under  $\Gamma_A$  by  $\hat{a} = \Gamma_A a$ . As there is no danger of confusion, instead of  $\Gamma_A A$  we will write  $\Gamma A$ , which is a unital (not necessarily closed) subalgebra of  $C(\Delta(A))$  separating the points of  $\Delta(A)$ . Recall too that  $r(a) = r(\hat{a}) = \|\hat{a}\|$  for all  $a \in A$ , and it is this fact that allows us to move from spectral isometries to isometries.

Let  $T: A \rightarrow B$  be a spectral isometry between the unital commutative semisimple Banach algebras  $A$  and  $B$ . We define  $\widehat{T}: \Gamma A \rightarrow \Gamma B$  by  $\widehat{T} = \Gamma_B \circ T \circ \Gamma_A^{-1}$ . Then  $\widehat{T}$  is a spectral isometry which is unital, or surjective, when  $T$  has these properties. Moreover, since spectral radius and norm coincide for continuous functions,  $\widehat{T}$  is in fact an isometry. Resulting from this observation, we can apply knowledge on isometries to gain information on spectral isometries, and our first application will be the following proposition.

**Proposition 2.3.** *Let  $T: A \rightarrow B$  be a spectral isometry between the unital semisimple Banach algebras  $A$  and  $B$ . If  $T$  is surjective then  $u = T1$  has its spectrum in the unit circle  $\mathbb{T}$ . If  $T$  is not necessarily surjective but  $B$  is commutative, the same conclusion holds.*

*Proof.* Let  $T_0: Z(A) \rightarrow B$  denote the restriction of  $T$  to the centre  $Z(A)$  of  $A$ . Clearly  $T_0$  is a spectral isometry which maps onto  $Z(B)$  if  $T$  is surjective [9, Proposition 4.3]. Applying the above transformation to  $T_0$  in this case, we obtain an isometry  $\widehat{T}_0: \Gamma Z(A) \rightarrow \Gamma Z(B)$  which is surjective. The function  $h \in C(\Delta(Z(B)))$  in Theorem 2.1, Equation (2.1) is nothing but  $\widehat{T}_0 1$  and has spectrum contained in  $\mathbb{T}$ . As  $u = T1 = \Gamma_B^{-1} \widehat{T}_0 \Gamma_A 1$  it follows that  $\sigma(u) \subseteq \mathbb{T}$  as claimed.

If  $T$  is not assumed to be surjective but  $B$  is commutative, we apply an analogous argument to obtain the same conclusion.  $\square$

As a consequence of this result, when studying surjective spectral isometries one can always reduce to the unital case. It is customary to call an element  $u$  in a Banach algebra

a *unitary* provided its spectrum  $\sigma(u)$  lies in  $\mathbb{T}$ . (This is because such  $u$  is invertible and  $\sigma(u^{-1}) \subseteq \mathbb{T}$  so  $u$  resembles a unitary operator on Hilbert space.)

**Corollary 2.4.** *Let  $T: A \rightarrow B$  be a surjective spectral isometry between the unital semisimple Banach algebras  $A$  and  $B$ . Then there are a unitary  $u \in Z(B)$  and a unital surjective spectral isometry  $T_1: A \rightarrow B$  such that*

$$Ta = u T_1 a \quad (a \in A). \quad (2.2)$$

*Proof.* Put  $u = T1$  which, by Proposition 2.3, is unitary and set  $T_1 a = u^{-1} Ta$ ,  $a \in A$ . Since  $u$  is central, for each  $a \in A$ ,

$$r(T_1 a) \leq r(u^{-1}) r(Ta) = r(Ta) = r(a) = r(uu^{-1} Ta) \leq r(u) r(u^{-1} Ta) = r(T_1 a)$$

whence  $T_1$  is a unital surjective spectral isometry.  $\square$

We also obtain a non-unital version of Nagasawa's theorem; see [2, Theorem 4.1.17].

**Corollary 2.5.** *Let  $T: A \rightarrow B$  be a surjective spectral isometry between the unital commutative semisimple Banach algebras  $A$  and  $B$ . Then there are a unitary  $u \in B$  and an algebra isomorphism  $T_1: A \rightarrow B$  such that*

$$Ta = u T_1 a \quad (a \in A). \quad (2.3)$$

*Proof.* The unital surjective spectral isometry  $T_1: A \rightarrow B$  given by Corollary 2.4 is an algebra isomorphism; either by Nagasawa's theorem or, more directly here, by the second part of Theorem 2.1 applied to the isometry  $\widehat{T}_1 = \Gamma_B T_1 \Gamma_A^{-1}$  as in the proof of Proposition 2.3.  $\square$

We shall now turn our attention to non-surjective spectral isometries with commutative codomain. By Proposition 2.3 we can focus on the case of a unital spectral isometry. As we saw above, even for a proper isometry the image of an algebra may not be an algebra so we need to analyse where the multiplicativity gets lost. Once again, Novinger's theorem (Theorem 2.1) will be our main tool as it describes the action of an isometry without the assumption of surjectivity.

Suppose that  $T: E \rightarrow C(Y)$  is a unital isometry defined on a *unital subalgebra*  $E$  of  $C(X)$ , where both  $X$  and  $Y$  are compact Hausdorff spaces. Suppose further that  $E$  separates the points of  $X$ . Throughout we will now denote the image of  $T$  by  $F = \text{im } T$  and we put  $Y_T = \overline{\text{ch}(F)}$ , the closure of the Choquet boundary of  $F$ . By (2.1) above, we have, for all  $f, g \in E$  and all  $t \in \text{ch}(F)$ ,

$$T(fg)(t) = (fg)(\varphi(t)) = f(\varphi(t)) g(\varphi(t)) = (Tf Tg)(t) \quad (2.4)$$

and hence, by continuity,  $T(fg)(t) = (Tf Tg)(t)$  for all  $t \in Y_T$ . It follows that  $T(fg) - Tf Tg$  is contained in the closed ideal  $I_T = \{k \in C(Y) \mid k(t) = 0 \text{ for all } t \in Y_T\}$  which is nothing but the kernel of the restriction homomorphism  $\rho_T: C(Y) \rightarrow C(Y_T)$ . Therefore the composition with  $T$  is multiplicative, and we have proved the following result.

**Proposition 2.6.** *Let  $X$  and  $Y$  be compact Hausdorff spaces. Let  $T$  be a unital isometry from a unital subalgebra  $E$  of  $C(X)$  which separates the points of  $X$  into  $C(Y)$ . With the above notation,  $\rho_T \circ T: E \rightarrow C(Y_T)$  is a unital algebra homomorphism.*

*Remark 2.7.* With the above notation and caveats suppose that  $F = \text{im } T$  separates the points of  $Y$ . Then  $Y_T$  coincides with the Shilov boundary  $\partial F$  of  $F$ ; cf. [6, Section 3.3].

By applying the Gelfand representation of commutative semisimple Banach algebras as before, we can immediately draw the following consequence for unital spectral isometries.

**Proposition 2.8.** *Let  $T: A \rightarrow B$  be a unital spectral isometry between the unital commutative semisimple Banach algebras  $A$  and  $B$ . Denote by  $\Delta_T$  the closure of the Choquet boundary of the image of  $\Gamma_B T$  in  $C(\Delta(B))$  and by  $\rho_T: C(\Delta(B)) \rightarrow C(\Delta_T)$  the restriction homomorphism. Then  $T_\rho = \rho_T \circ \Gamma_B \circ T$  is a unital algebra homomorphism from  $A$  into  $C(\Delta_T)$ .*

*Proof.* As in the proof of Proposition 2.3 we define  $\hat{T} = \Gamma_B \circ T \circ \Gamma_A^{-1}$  and obtain a unital isometry from  $\Gamma A \subseteq C(\Delta(A))$  onto  $\text{im } \Gamma_B T \subseteq C(\Delta(B))$ . By Proposition 2.6,  $\rho_{\hat{T}} \circ \hat{T}$  is multiplicative from  $\Gamma A$  into  $C(\Delta(B)_{\hat{T}})$ . For all  $x, y \in A$  we thus obtain

$$\begin{aligned} T_\rho(xy) &= \rho_T \circ \Gamma_B \circ T(xy) = \rho_{\hat{T}} \circ \hat{T} \circ \Gamma_A(xy) \\ &= \rho_{\hat{T}} \circ \hat{T}(\hat{x} \hat{y}) = \rho_{\hat{T}} \circ \hat{T}(\hat{x}) \rho_{\hat{T}} \circ \hat{T}(\hat{y}) \\ &= \rho_{\hat{T}}(\hat{T}(\hat{x}) \hat{T}(\hat{y})) = \rho_T(\Gamma_B \circ T(x) \Gamma_B \circ T(y)) \\ &= \rho_T \circ \Gamma_B \circ T(x) \rho_T \circ \Gamma_B \circ T(y) = T_\rho(x) T_\rho(y) \end{aligned}$$

which proves the claim.  $\square$

Finally, putting everything together, we obtain our main result.

**Theorem 2.9.** *Let  $T: A \rightarrow B$  be a spectral isometry between the unital semisimple Banach algebras  $A$  and  $B$  and suppose that  $B$  is commutative. Then  $T_1$  is unitary in  $B$  and, with  $v = \rho_T(\Gamma_B T_1)^{-1}$ , the mapping  $a \mapsto v \rho_T(\Gamma_B T a)$  is multiplicative from  $A$  into  $C(\Delta_T)$ , where  $\Delta_T = \overline{\text{ch}(\text{im } \Gamma_B T)}$  and  $\rho_T$  denotes the restriction mapping.*

*Proof.* By Proposition 2.3,  $T_1$  is a unitary in  $B$  and therefore  $T_1$  defined by  $T_1 a = u^{-1} T a$ ,  $a \in A$  is a unital spectral isometry into  $B$ . Composing  $T_1$  with  $\Gamma_B$  we obtain a unital spectral isometry into  $C(\Delta(B))$ . By [8, Lemma 2.1],  $S = \Gamma_B \circ T_1$  is a trace, that is,  $S(xy) = S(yx)$  for all  $x, y \in A$ . Since  $B$  is semisimple,  $\Gamma_B$  is injective, and since  $A$  is semisimple,  $T_1$  is injective [9, Proposition 4.2]. As a result,  $A$  is commutative and we can apply Proposition 2.8 to  $T_1$ . Note that  $\text{im } \Gamma_B T_1$  and  $\text{im } \Gamma_B T$  have the same Choquet boundary in  $\Delta(B)$  since multiplication by  $\Gamma_B(u)$  is an isometric bijection between these spaces so their dual spaces are isometrically isomorphic. Therefore  $\rho_T \circ \Gamma_B \circ T_1$  is a unital algebra homomorphism from  $A$  into  $C(\Delta_T)$ . Finally the identity

$$\rho_T \circ \Gamma_B \circ T_1(a) = \rho_T \circ \Gamma_B \circ (u^{-1} T a) = v \rho_T(\Gamma_B T a) \quad (a \in A)$$

completes the proof.  $\square$

## REFERENCES

- [1] J. Alaminos, J. Extremera and A. R. Villena, *Spectral preservers and approximate spectral preservers on operator algebras*, J. Funct. Anal., to appear.
- [2] B. Aupetit, *A primer on spectral theory*, Springer-Verlag, New York, 1991.
- [3] C. Costara and D. Repovš, *Spectral isometries onto algebras having a separating family of finite-dimensional irreducible representations*, J. Math. Anal. Appl. **365** (2010), 605–608.
- [4] R. J. Fleming and J. J. Jamison, *Isometries on Banach spaces: function algebras*, Monographs and Surveys in Pure and Appl. Maths. **129**, Chapman and Hall, Boca Raton, 2003.
- [5] R. V. Kadison, *Isometries of operator algebras*, Annals of Math. **54** (1951), 325–338.
- [6] E. Kaniuth, *A course in commutative Banach algebras*, Grad. Texts in Maths. **246**, Springer-Verlag, New York, 2009.
- [7] M. Mathieu, *Towards a non-selfadjoint version of Kadison’s theorem*, Ann. Math. Inf. **32** (2005), 87–94.
- [8] M. Mathieu, *Spectrally bounded traces on  $C^*$ -algebras*, Bull. Austral. Math. Soc. **68** (2003), 169–173.
- [9] M. Mathieu and G. J. Schick, *First results on spectrally bounded operators*, Studia Math. **152** (2002), 187–199.
- [10] M. Mathieu and A. R. Sourour, *Spectral isometries on non-simple  $C^*$ -algebras*, Proc. Amer. Math. Soc. **142** (2014), 129–145.

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